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Algebraic inequalities with polynomial $2(xy+yz+xz)-(x^2+y^2+z^2)$

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Part 1. As introduction.

1.1. Symmetric polynomial

$$\Delta(x,y,z) := 2(xy+yz+xz)-(x^2+y^2+z^2)$$

is quadratic form, which isn't positive definite

$$\Delta\left(p+q, \frac{q+r}{2}, \frac{q-r}{2}\right) = q^2-p^2-r^2$$

and even claim $x,y,z>0$ don't provide positivity of $\Delta(x,y,z)$.But if a,b,c be sidelengths of a triangle then $\Delta(a,b,c)>0$ and in that case importance of $\Delta(a,b,c)$ has been demonstrated in **[1]**.
In that paper we are going to pay attention only on to the algebraic inequalities with polynomial $\Delta(a,b,c)$ where $a,b,c>0$, mainly to the inequalities which in certain sense generalize Hadwiger-Finsler Inequality

$$4\sqrt{3}F+(a-b)^2+(b-c)^2+(c-a)^2 \leq a^2+b^2+c^2.$$

Since

$$16F^2 = 2\sum a^2b^2 - \sum a^4 = \Delta\left(a^2,b^2,c^2\right)$$

and

$$\sum a^2 - \sum (a-b)^2 = \Delta(a,b,c)$$

(here and everywhere further $\sum$ is cyclic sum $\sum\limits_{cyc}$) then this inequality in Δ notation, brought before us in the form of inequality

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(H-F)

$$\Delta^2(a,b,c) \geq 3\Delta\left(a^2,b^2,c^2\right),$$

which holds not only for sidelengths a,b,c but also for any real positive a,b,c as well.
Indeed, assuming $a+b+c=1$ and denoting $p:=ab+bc+ca, q:=abc$ ($(1,p,q)-$notation) we obtain
$\Delta(a,b,c)=4p-1, \Delta\left(a^2,b^2,c^2\right)=4p-8q-1$ and then inequality (H-F) becomes

$$(4p-1)^2 \geq 3(4p-8q-1) \iff q \geq \frac{(1-p)(4p-1)}{6},$$

where latter inequality is Schure's Inequality $\sum\limits_{cyc} a^2(a-b)(a-c) \geq 0$ in $p,q-$notation, normalized by $\sum a=1$.
Using notation $\Delta_p(x,y,z):=\Delta(x^p,y^p,z^p)$, $p\in\mathbb{R}_+$, we can rewrite inequality (H-F) as

$$\Delta_1^2(a,b,c) \geq 3\Delta_2(a,b,c) = \Delta_0(a,b,c)\cdot\Delta_2(a,b,c).$$

It was also proved that in case a,b,c be sidelengths of a triangle holds inequality

$$\Delta_1(a,b,c)\cdot\Delta_3(a,b,c) \geq 3\ \Delta_4(a,b,c) \qquad [3]$$

and for any positive real a,b,c holds inequality

$$\Delta_2^2(a,b,c) \geq \Delta_1(a,b,c)\cdot\Delta_3(a,b,c). \qquad [4]$$

It was the strong argument for attempt to prove generalization of such inequalities
for any positive real $a,b,c,$ namely, inequality

$$\Delta_1(a,b,c)\cdot\Delta_n(a,b,c) \geq 3\ \Delta_{n+1}(a,b,c)$$

and even more general inequality

$$\Delta_p(a,b,c)\cdot\Delta_q(a,b,c) \geq 3\ \Delta_{p+q}(a,b,c), p,q\in\mathbb{R}_+ \tag{1}$$

(which in turn is a consequence
of the more general inequality (3) represented by Theorem in Part 2) and inequality

$$\Delta_n^2(a,b,c) \geq \Delta_{n-1}(a,b,c)\cdot\Delta_{n+1}(a,b,c) \quad ,n\in\mathbb{N} \tag{2}$$

which can be considered as strong generalization of Hadwiger-Finsler Inequality
and the proof of which is the topic of a separate article.

1.2. As more deep introduction in Δ we consider two propositions:

Proposition 1. Let $a,b,c>0$. Then $\Delta_n(a,b,c)>0$ for any $n\in\mathbb{N}$ iff $a=b\geq c$ or cyclic more two variants.
Proof. Note that

$$\Delta\left(x^2,y^2,z^2\right)=(x+y+z)(x+y-z)(x-y+z)(-x+y+z)$$

and for positive x,y,z we have equivalency

$$\Delta\left(x^2,y^2,z^2\right)>0 \iff \begin{cases} x+y>z \\ y+z>x \\ z+x>y \end{cases}.$$

Due symmetry and homogeneity of $\Delta_n(a,b,c)>0$ WLOG we assume that $a\geq b\geq 1$.
Then for any even $n\in\mathbb{N}$ we have

$$\begin{cases} \Delta_{2n}(a,b,c)>0 \\ a\geq b\geq c=1 \end{cases} \iff \begin{cases} b^n+1>a^n \\ a\geq b\geq c=1 \end{cases}$$

First we have to make two comments in relation to $\Delta_n(a,b,c)$.
Suppose that $a>b$, then

$$a^n=(b+(a-b))^n>b^n+n(a-b)b^{n-1}>b^n+n(a-b)>b^n+1$$

for any $n>\dfrac{1}{a-b}$. It is contradict to $b^n+1>a^n$ which holds for any $n\in\mathbb{N}$.Thus $a=b$.
Let now $a=b\geq c$ then

$$\Delta(a^n,b^n,c^n)=2a^{\frac{n}{2}}b^{\frac{n}{2}}+2b^{\frac{n}{2}}c^{\frac{n}{2}}+2c^{\frac{n}{2}}a^{\frac{n}{2}}-a^n-b^n-c^n=$$

$$=4c^{\frac{n}{2}}a^{\frac{n}{2}}-c^n\geq 3c^n>0$$

Further where this does not lead to confusion, we shall write Δ_n instead $\Delta_n(a,b,c)$.

Proposition 2. If for real $a, b, c, p > 0$ holds $\Delta_{2p}(a, b, c) < 0$ then for any real $q > p$ holds $\Delta_{2q}(a, b, c) < 0$.

Proof. Since

$$\Delta_{2p} = (a^p + b^p + c^p)(a^p + b^p - c^p)(a^p - b^p + c^p)(-a^p + b^p + c^p)$$

and $a, b, c > 0$ then inequality $\Delta_{2p} < 0$ mean that exactly one of three factors $a^p + b^p - c^p, a^p - b^p + c^p, -a^p + b^p + c^p$ is negative, let it be ${}^p + b^p - c^p$. Since $a^p + b^p < c^p$ then

$$\left(\frac{a}{c}\right)^q + \left(\frac{b}{c}\right)^q < \left(\frac{a}{c}\right)^p + \left(\frac{b}{c}\right)^p < 1,$$

because $\frac{a}{c}, \frac{b}{c} < 1$ yields

$$\left(\frac{a}{c}\right)^q < \left(\frac{a}{c}\right)^p, \left(\frac{b}{c}\right)^q < \left(\frac{b}{c}\right)^p.$$

Thus, $a^q + b^q - c^q < 0$ and since that yields $c^q > a^q, b^q$ then

$$a^p - b^p + c^p > 0, -a^p + b^p + c^p > 0$$

and, therefore, $\Delta_{2q} < 0$.

Corollary. If $\Delta_p(a, b, c) \geq 0$ then for any $0 < q < p$ holds $\Delta_q(a, b, c) \geq 0$.

Part 2. Generalizations of inequalities (1) and (2)

2.1. (related to inequality (1))

Theorem. For any two triples (a, b, c) and (x, y, z) of nonnegative real numbers which agreed upon in order ($(a - b)(x - y) \geq 0$ and cyclic

$$(b - c)(y - z) \geq 0, (c - a)(z - x) \geq 0$$

holds inequality

$$\Delta(a, b, c)\, \Delta(x, y, z) \geq 3\Delta(ax, by, cz) \qquad (3)$$

Proof. Since inequality **(3)** is invariant with respect to permutations of $((a,x),(b,y),(c,z))$ we assume that $a \ge b \ge c, x \ge y \ge z$. Also, since **(3)** is invariant with respect to simultaneous transposition a with x, b with y and c with z we may assume that

$$(a-b)(y-z) \ge (x-y)(b-c),$$

because, in the case $(a-b)(y-z) < (x-y)(b-c)$, by such transposition we obtain

$$(x-y)(b-c) < (a-b)(y-z).$$

Since $a \ge b \ge c,\ x \ge y \ge z$ we have

$$(a-b)(y-z) \ge (x-y)(b-c) \iff (a-b)(x-z) \ge (x-y)(a-c) \Leftrightarrow$$

$$\iff (a-c)(y-z) \ge (b-c)(x-z) \iff ay+bz+cx \ge bx+cy+az.$$

Indeed,

$$(a-b)(y-z)-(x-y)(b-c) = (a-b)(x-z)-(x-y)(a-c) =$$

$$= (a-c)(y-z)-(b-c)(x-z) = ay+bz+cx-(bx+az+cy)$$

First, we exclude from consideration the cases when at least two of the variables x, y, z are equal. If $x=y=z$ then $\Delta(x,y,z) = 3x^2$ and $\Delta(ax,by,cz) = x^2\Delta(a,b,c)$ and we obtain

$$\Delta(a,b,c)\,\Delta(x,y,z) = 3\Delta(ax,by,cz).$$

Thus, remains consider cases $x > y = z$ and $x = y > z$.
If $x > y = z$ then $0 = (a-b)(y-z) \ge (x-y)(b-c) \ge 0 \implies b = c$ and we obtain

$$\Delta(a,b,c) = 4ab-a^2, \Delta(x,y,z) = 4xy-x^2, \Delta(ax,by,cz) =$$

$$= \Delta(ax,by,by) = 4axby - a^2x^2$$

and inequality **(3)** become

$$\left(4ab-a^2\right)\left(4xy-x^2\right)\geq 3\left(4axby-a^2x^2\right)\iff$$

$$\Leftrightarrow\left(4ab-a^2\right)\left(4xy-x^2\right)-3\left(4axby-a^2x^2\right)\geq 0\iff 4ax\left(x-y\right)\left(a-b\right)\geq 0$$

If $x=y>z$ then

$$\Delta\left(a,b,c\right)\Delta\left(x,y,z\right)-3\Delta\left(ax,by,cz\right)=$$

$$=\left(2ab+2bc+2ca-a^2-b^2-c^2\right)\left(4xz-z^2\right)-$$

$$-3\left(2cxz\left(a+b\right)-x^2\left(a-b\right)^2-c^2z^2\right)=$$

$$=\left(x-z\right)\left(\left(3x-z\right)\left(a-b\right)^2+zc\left(2\left(a+b\right)-4c\right)\right)\geq 0$$

So, from now we can assume that $x>y>z$ and, therefore,

$$\left(a-b\right)\left(y-z\right)\geq\left(x-y\right)\left(b-c\right)\iff ay+bz+cx\geq bx+cy+az\Leftrightarrow$$

$$\iff\frac{b-c}{y-z}\leq\frac{a-c}{x-z}\leq\frac{a-b}{x-y}. \tag{4}$$

On the next step we will represent $\Delta\left(a,b,c\right)\Delta\left(x,y,z\right)-3\Delta\left(ax,by,cz\right)$ in the form

$$u\left(y-z\right)^2+v\left(z-x\right)^2+w\left(x-y\right)^2,$$

convenient for application of SOS Method, more precisely for it's following modification:
Let (x,y,z) and (u,v,w) be triples of real numbers such that $x\geq y\geq z$ and $u\geq v\geq w, v+w\geq 0$ (or $u\leq v\leq w, u+v\geq 0$). Then

$$u\left(y-z\right)^2+v\left(z-x\right)^2+w\left(x-y\right)^2\geq 0,$$

which immediately follow from original form of SOS Theorem 2 "If $x\geq y\geq z$ and numbers $u,v,\ w$ satisfy to inequalities $u+v\geq 0, v+w\geq 0$ and $v\geq 0$ then

$$u\left(y-z\right)^2+v\left(z-x\right)^2+w\left(x-y\right)^2\geq 0"$$

because $u \geq v \geq w$ yield $u+v \geq v+w \geq 0$ and $2v \geq v+w \geq 0 \implies v \geq 0$.
To simplify the procedure of transforming we denote $P=bc, Q=ca, R=ab$, $A=a^2$, $B=b^2$, $C=c^2$, $P_1=yz$, $Q_1=zx$, $R_1=xy$, $A_1=x^2$, $B_1=y^2$, $C_1=z^2$ and will use $\sum$ instead cyclic sum $\sum_{cyc}$.
We have

$$\Delta(a,b,c)\Delta(x,y,z)-3\Delta(ax,by,cz)=\sum\left(2bc-a^2\right)\cdot\sum\left(2yz-x^2\right)-$$

$$-3\sum\left(2bycz-a^2x^2\right)=\left(2\sum P-\sum A\right)\left(2\sum P_1-\sum A_1\right)-$$

$$-3\left(2\sum PP_1-\sum AA_1\right)=$$

$$=4\sum P\sum P_1+\sum A\sum A_1-2\sum A_1\sum P-2\sum A\sum P_1-$$

$$6\sum PP_1+3\sum AA_1=$$

$$=2\sum P\sum P_1+2\sum A\sum A_1-2\sum A_1\sum P-2\sum A\sum P_1-$$

$$-6\sum PP_1+3\sum AA_1+2\sum P\sum P_1-\sum A\sum A_1=$$

$$=2\sum(A-P)\sum(A_1-P_1)+$$

$$+\left(3\sum AA_1-\sum A\sum A_1\right)-2\left(\sum 3PP_1-\sum P\sum P_1\right)=$$

$$=2\sum(A-P)\sum(A_1-P_1)+$$

$$+\sum(2A_1-B_1-C_1)A-2\sum(2P_1-Q_1-R_1)P.$$

Since

$$\sum(A-P)=\frac{1}{2}\left(\sum(a-b)^2\right),$$

$$\sum(A_1-P_1)=\frac{1}{2}\left(\sum(x-y)^2\right),$$

$$(2A_1 - B_1 - C_1)A = \sum (A - B)(A_1 - B_1) = \sum (a^2 - b^2)(x^2 - y^2) =$$

$$= \sum \frac{(a^2 - b^2)(x + y)}{x - y}(x - y)^2$$

and

$$\sum (2P_1 - Q_1 - R_1) P =$$

$$= \sum (P - Q)(P_1 - Q_1) = \sum (bc - ca)(yz - zx) = \sum \frac{cz(a - b)}{x - y}(x - y)^2$$

then

$$\Delta(a, b, c)\,\Delta(x, y, z) - 3\Delta(ax, by, cz) = w(x - y)^2 + u(y - z)^2 + v(z - x)^2$$

where

$$w = \frac{1}{2}\sum (a - b)^2 + \frac{(a^2 - b^2)(x + y)}{x - y} - \frac{2cz(a - b)}{x - y} =$$

$$= D + \frac{a - b}{x - y}((a + b)(x + y) - 2cz)$$

$$v = \frac{1}{2}\sum (c - a)^2 + \frac{(c^2 - a^2)(z + x)}{z - x} - \frac{2by(c - a)}{z - x} =$$

$$= D + \frac{a - c}{x - z}((a + b)(x + y) - 2by)$$

$$u = \frac{1}{2}\sum (b - c)^2 + \frac{(b^2 - c^2)(y + z)}{y - z} - \frac{2ax(b - c)}{y - z} =$$

$$= D + \frac{b - c}{y - z}((a + b)(x + y) - 2ax)$$

and

$$D = \frac{1}{2}\sum (a - b)^2$$

Thus, to prove inequality $\Delta(a,b,c)\,\Delta(x,y,z)\geq 3\Delta(ax,by,cz)$ suffice to prove that $w\geq v\geq u$ and $v+w\geq 0$ using $a\geq b\geq c, x>y>z$ and (4).
In order to obtain the inequality $w\geq v\geq u$ we have to prove inequalities

$$\frac{b-c}{y-z}\left((b+c)(y+z)-2ax\right)\leq\frac{a-c}{x-z}\left((c+a)(z+x)-2by\right)\leq$$

$$\leq\frac{a-b}{x-y}\left((a+b)(x+y)-2cz\right).$$

Since $\dfrac{b-c}{y-z}\leq\dfrac{a-c}{x-z}\leq\dfrac{a-b}{x-y}$ then remains to prove

$$(b+c)(y+z)-2ax\leq(c+a)(z+x)-2by\leq(a+b)(x+y)-2cz.$$

We have

$$(c+a)(z+x)-2by-\left((b+c)(y+z)-2ax\right)=$$

$$=3(ax-by)+z(a-b)+c(x-y)\geq 0$$

and

$$(a+b)(x+y)-2cz-\left((c+a)(z+x)-2by\right)=$$

$$=3(by-cz)+a(y-z)+x(b-c)\geq 0.$$

And finally we will prove inequality $u+v\geq 0$.
We have

$$u+v-(a-b)^2=\left((b-c)^2+\frac{b-c}{y-z}\left((a+b)(x+y)-2ax\right)\right)+$$

$$+\left((a-c)^2+\frac{a-c}{x-z}\left((a+b)(x+y)-2by\right)\right)=$$

$$=\frac{b-c}{y-z}\left((b-c)(y-z)+(b+c)(y+z)-2ax\right)+$$

$$\frac{a-c}{x-z}\left((a-c)(x-z)+(a+c)(x+z)-2by\right)=\frac{2(b-c)}{y-z}(by+cz-ax)+$$

$$+\frac{2(a-c)}{x-z}(ax+cz-by)\;=$$

$$= \left(2cz\left(\frac{b-c}{y-z}+\frac{a-c}{x-z}\right)+2\left(ax-by\right)\left(\frac{a-c}{x-z}-\frac{b-c}{y-z}\right)\right) \geq 0$$

Corollary. For any positive real numbers a, b, c, p and q holds inequality

$$\Delta\left(a^p, b^p, c^p\right)\Delta\left(a^q, b^q, c^q\right) \geq 3\Delta\left(a^{p+q}, b^{p+q}, c^{p+q}\right).$$

Proof. By replacing in inequality **(3)** (a, b, c) and (x, y, z) with (a^p, b^p, c^p) and (a^q, b^q, c^q), respectively,we obtain inequality

$$\Delta\left(a^p, b^p, c^p\right)\Delta\left(a^q, b^q, c^q\right) \geq 3\Delta\left(a^{p+q}, b^{p+q}, c^{p+q}\right).$$

2.2. Remark.(related to inequality (2)). *In supposition* $\Delta_2 > 0$ *and using inequalities* $\Delta_1^2 \geq 3\Delta_2, \Delta_2^2 \geq \Delta_2\Delta_3$ *and* $\Delta_3^2 \geq \Delta_2\Delta_4$ *we obtain the following chain of inequalities*

$$\frac{\Delta_1}{3} \geq \sqrt{\frac{\Delta_2}{3}} \geq \sqrt[3]{\frac{\Delta_3}{3}}.$$

Indeed, $\Delta_2 > 0 \implies \Delta_1 > 0$ and then

$$\Delta_1^2 \geq 3\Delta_2 \iff \frac{\Delta_1}{3} \geq \sqrt{\frac{\Delta_2}{3}}.$$

Also we have

$$\Delta_2^4 \geq \Delta_2^2\Delta_3^2 \geq 3\Delta_2\Delta_3^2 \implies \Delta_2^4 \geq 3\Delta_2\Delta_3^2 \implies \Delta_2^3 \geq$$

$$\geq 3\Delta_3^2 \iff \sqrt{\frac{\Delta_2}{3}} \geq \sqrt[3]{\frac{\Delta_3}{3}}.$$

Similarly, in supposition $\Delta_n(a, b, c) > 0$, using inequality **(2)** and Math Induction can be proved chain of inequalities

$$\frac{\Delta_1}{3} \geq \sqrt{\frac{\Delta_2}{3}} \geq \cdots \geq \sqrt[n]{\frac{\Delta_n}{3}}.$$

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